Newton-Raphson Method For Solving Nonlinear Equations
Method Basics

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Linear equations such as $2x = 4$ are easy to solve. We know when an equation is linear because its first derivative is a constant, which means it has a constant slope (i.e. is linear). Linear equations like this one can be solved using elementary mathematical operations such as addition, subtraction, multiplication and/or division. Non-linear equations are equations where the first derivative is not a constant. The equation $x^3 = 8$ is non-linear because its first derivative $(3\,x^2)$ is not a constant in that it depends on the value of x. Simple non-linear equations like this one can be solved by taking the third root of x or by using natural logarithms, both of which are buttons on a calculator or included in software packages such as Excel. To the serious student of finance, these easy-to-solve equations are a luxury. Consider the following problem in finance...

Our Hypothetical Problem

Start-up company ABC is a typical start-up in that it's revenue growth rate is initially very high (because it's current revenue base and market share are small) but it's fixed costs are such that it is incurring losses. The goal of this company is to grow it's revenue base and become profitable before it runs out of cash. We calculate that the company must take in \$50 million in gross revenue over the next three years or else it will fail. Its current annualized revenue base is \$10 million. The question becomes "What does the revenue growth rate have to average over the next three years for this company to survive?"

Legend of Symbols:

a	=	Cumulative gross revenue target (\$50 million)
b	=	Current annualized revenue base (\$10 million)
r	=	Average revenue growth rate
t	=	How much time we have in years

Equation (1) is the break-even equation and the unknown that we have to solve for is r, the average revenue growth rate.

$$b \times \int_0^t \mathrm{Exp}\left\{r\,t\right\}\delta t = a \tag{1}$$

We will define the function $f(r)$ to be the break-even equation. When we solve the integral in equation (1) and replace the known variables with their actual values the break-even equation becomes...

$$f(r) = 10\left(\mathrm{Exp}\left\{3\,r\right\} - 1\right) r^{-1} = 50 \tag{2}$$

Notice that equation (2) is non-linear and that the unknown to solve for (r) appears twice in the equation - once as the denominator of a fraction and once as an exponent. We could solve for r by manually plugging values into the equation until we arrive at the solution, or we could plug the above equation into Excel and use Excel's Solver tool. On the other hand, we could utilize the Newton-Raphson method for solving non-linear univariate equations...

Building Our Model

We are given the function $f(x)$ and the equations $f_x(x)$, $f_{xx}(x)$, $f_{xxx}(x)$, and so on, which are the first, second, third, and so on, derivatives of the dependent variable $f(x)$ with respect to the independent variable x. Using a

Taylor Series Expansion the equation for $f(x)$ as a function of $f(\hat{x})$ where $x \neq \hat{x}$ is... [?]

$$f(x) = f(\hat{x}) + f_x(\hat{x})\,(x-\hat{x}) + \frac{1}{2}\,f_{xx}(\hat{x})\,(x-\hat{x})^2 + \frac{1}{6}\,f_{xxx}(\hat{x})\,(x-\hat{x})^3 + ... \tag{3}$$

If we drop the second order and higher derivatives in Equation (7) above then that equation can me rewriten as...

$$f(x) = f(\hat{x}) + f_x(\hat{x})\,(x-\hat{x}) + \epsilon \;\text{ ...where... }\; \epsilon = \text{error term} \tag{4}$$

Assume that we are given the value of the dependent variable $f(x)$ and are asked to solve for the independent variable x. We start by isolating the variable x in Equation (8) above to one side of that equation. Note that we can rewrite Equation (8) above as...

$$x = \hat{x} + \frac{f(x) - f(\hat{x}) - \epsilon}{f_x(\hat{x})} \tag{5}$$

If we move the error term to the left hand side of the equation then Equation (9) above becomes...

$$x + \hat{\epsilon} = \hat{x} + \frac{f(x) - f(\hat{x})}{f_x(\hat{x})} \;\text{ ...where... }\; \hat{\epsilon} = \frac{\epsilon}{f_x(\hat{x})} \tag{6}$$

Given that the value of $f(x)$ is known, to solve for the variable x we will iterate Equation (**??**) above as follows...

Step	Task Description
1	Define the guess variable $\hat{x}$ such that $f(\hat{x})$ is close to $f(x)$ (the closer it is the less iterations).
2	Plug guess variable $\hat{x}$ into Equation (**??**) above and recalculate $x + \hat{\epsilon}$.
3	Using the results from Step 2 above redefine the guess variable to be $\hat{x} = x + \hat{\epsilon}$.
4	Go to Step 2 above and repeat.

If after each iteration the equation $|\,f(x)$ - $f(\hat{x})\,|$ gets closer and closer to zero then the process will converge to the true value of x.

The Answer To Our Hypothetical Problem

Using Equations (2) above and Appendix Equation (13) below the equations needed to solve our problem are...

$$f(r) = 50 \;\text{ ...and... }\; f(\hat{r}) = 10\left(\text{Exp}\left\{3\,\hat{r}\right\} - 1\right)\hat{r}^{-1} \;\text{ ...and... }\; f_r(\hat{r}) = 30\,\text{Exp}\left\{3\,\hat{r}\right\}\hat{r}^{-1} - 10\left(\text{Exp}\left\{3\,\hat{r}\right\} - 1\right)\hat{r}^{-2} \tag{7}$$

If we start with a guess value of $\hat{r} = 0.40$ then using Equations (**??**) above and iterating Equation (**??**) above then the answer to our problem is...

Iteration	$r + \hat{\epsilon}$	$\hat{r}$	$f(r)$	$f(\hat{r})$	$f_r(\hat{r})$
1	0.32305	0.40000	50.00	58.00	104.00
2	0.31586	0.32305	50.00	50.63	88.03
3	0.31580	0.31586	50.00	50.00	86.68
4	0.31580	0.31580	50.00	50.00	86.67

Per the table above the required annual continuous-time growth rate = 31.58%. Note that it took only four iterations to calculate our value [at iteration #4 $|\,f(r) - f(\hat{r})\,| = 0$].

Appendix

A. The equation for the derivative of Equation (2) above is...

$$\text{if... } \theta_1 = \text{Exp}\left\{3\,\hat{r}\right\} - 1 \;\text{ ...and... }\; \theta_2 = \hat{r}^{-1} \;\text{ ...then... }\; f(\hat{r}) = 10\,\theta_1\theta_2 \;\text{ ...and... }\; f'(\hat{r}) = 10\left(\frac{\delta\theta_1}{\delta\hat{r}}\theta_2 + \frac{\delta\theta_2}{\delta\hat{r}}\theta_1\right) \tag{8}$$

Using Equation (12) above...

$$\text{if... } \frac{\delta\theta_1}{\delta\hat{r}} = 3\,\text{Exp}\left\{3\,\hat{r}\right\} \;\text{ ...and... }\; \frac{\delta\theta_2}{\delta\hat{r}} = -\hat{r}^{-2} \;\text{ ...then... }\; f'(\hat{r}) = 30\,\text{Exp}\left\{3\,\hat{r}\right\}\hat{r}^{-1} - 10\left(\text{Exp}\left\{3\,\hat{r}\right\} - 1\right)\hat{r}^{-2} \tag{9}$$

References

[1] Gary Schurman, *The Taylor Series Expansion,* November, 2017.